Imperial College London

Lecture 6

Windowing Effects & Discrete Fourier Transform

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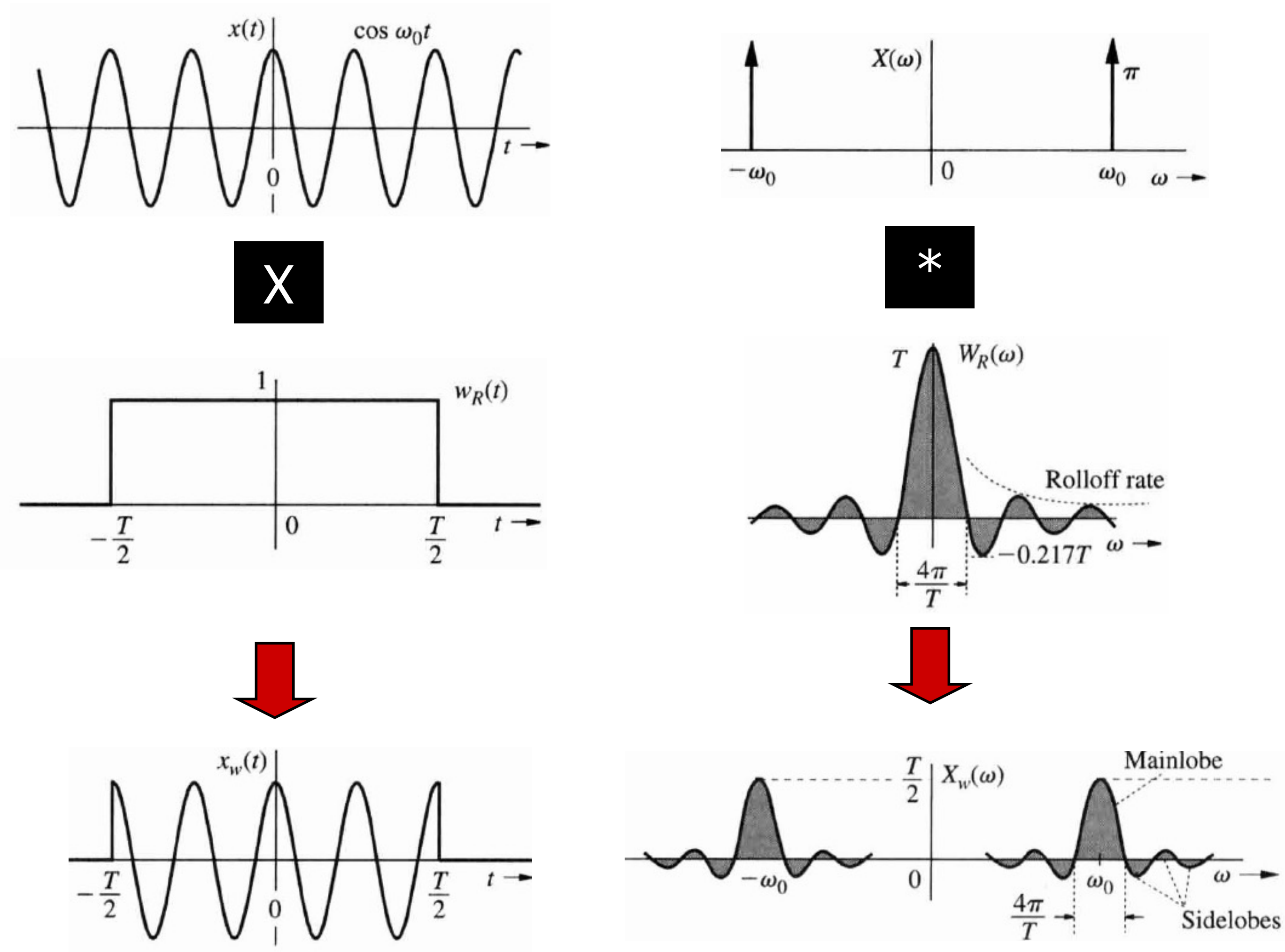
Windowing and its effect

- Extracting a segment of a signal in time is the same as multiplying the signal with a rectangular window:



Spectral spreading

Energy spread out from ω0 to width of 2π/T – reduced spectral resolution.

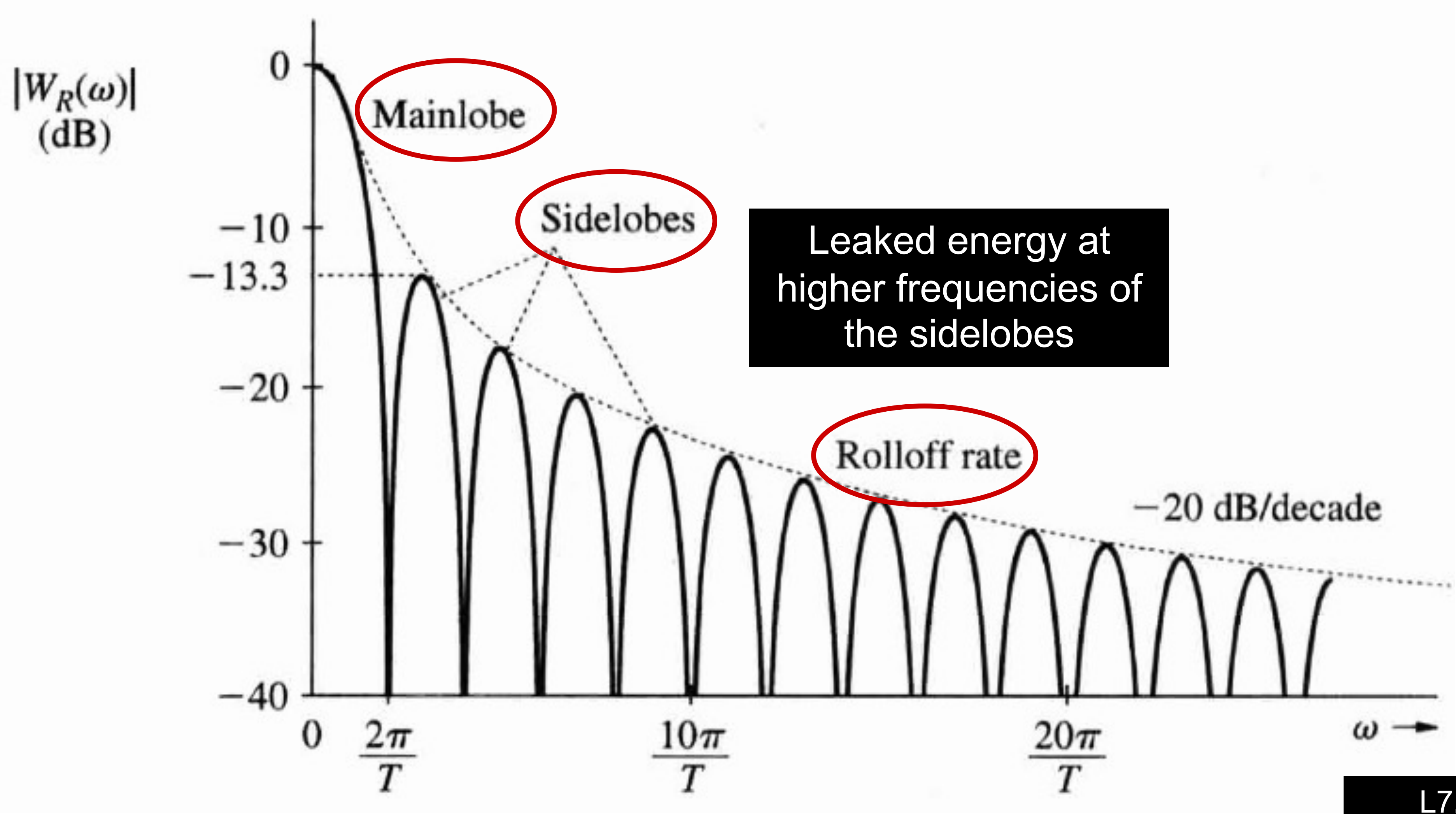
Leakage

Energy leaks out from the mainlobe to the sidelobes.

L7.8

Mainlobe & Sidelobes in dB

- Detail effects of windowing (rectangular window):



Leaked energy at higher frequencies of the sidelobes

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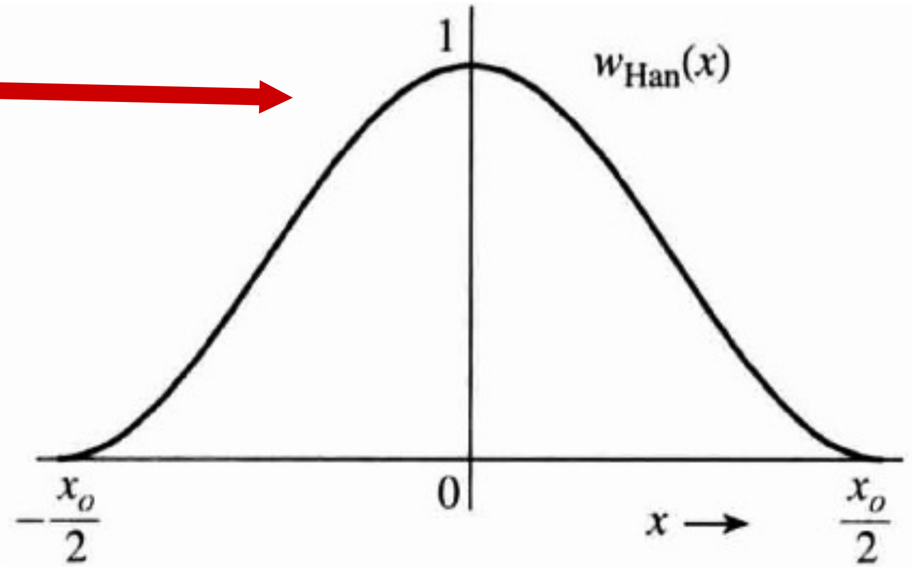
Remedies for side effects of truncation

1. Make mainlobe width as narrow as possible ➔ implies as wide a window as possible.
2. Avoid big discontinuity in the windowing function to reduce leakage (i.e. high frequency sidelobes).
3. 1) and 2) above are incompatible – therefore needs a compromise.

◆ Commonly replace rectangular window with one of these:
   - Hamming window
   - Hanning window
   - Barlett window
   - Blackman window
   - Kaiser window



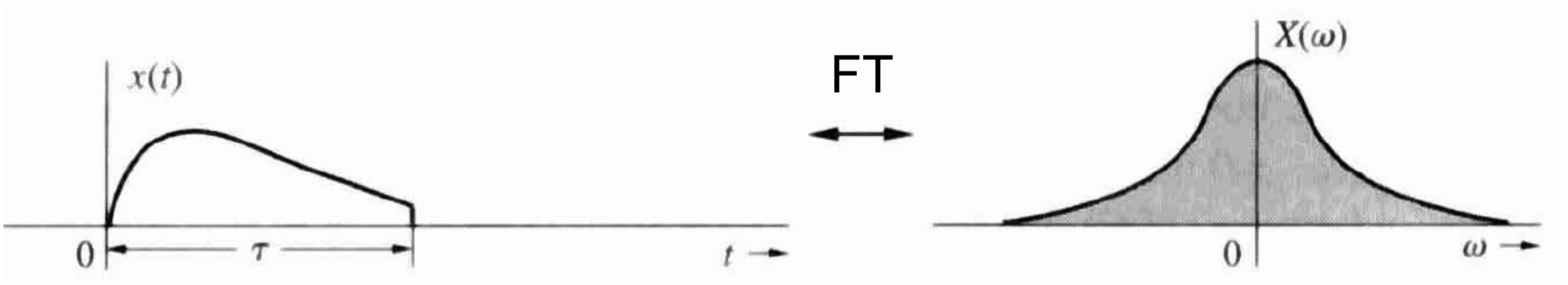
L7.8

Comparison of different windowing functions

No.	Window $w(t)$	Mainlobe Width	Rolloff Rate (dB/oct)	Peak Sidelobe level (dB)
1	Rectangular: $\text{rect}\left(\frac{t}{T}\right)$	$\frac{4\pi}{T}$	-6	-13.3
2	Bartlett: $\Delta\left(\frac{t}{2T}\right)$	$\frac{8\pi}{T}$	-12	-26.5
3	Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$	$\frac{8\pi}{T}$	-18	-31.5
4	Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$	$\frac{8\pi}{T}$	-6	-42.7
5	Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$	$\frac{12\pi}{T}$	-18	-58.1
6	Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{t}{T}\right)^2}\right]}{I_0(\alpha)}$ $0 \leq \alpha \leq 10$	$\frac{11.2\pi}{T}$	-6	-59.9 ($\alpha = 8.168$)

Spectral Sampling (1)

- As expected, time domain sampling has a dual: spectral sampling.
- Consider a time limited signal x(t) with a spectrum X(ω).

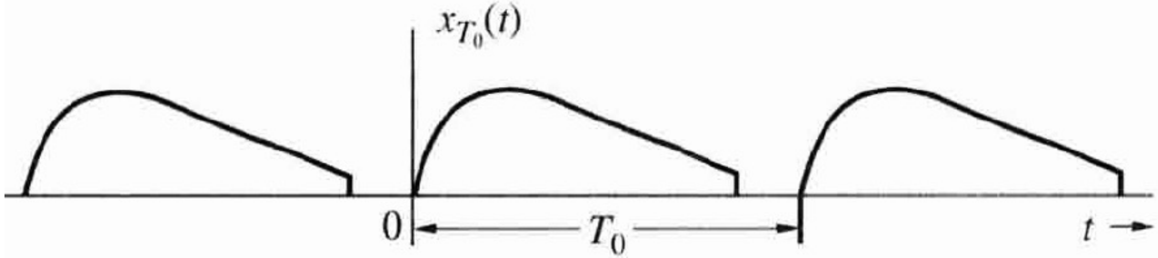


$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad = \int_{0}^{\tau} x(t)e^{-j\omega t}dt$$

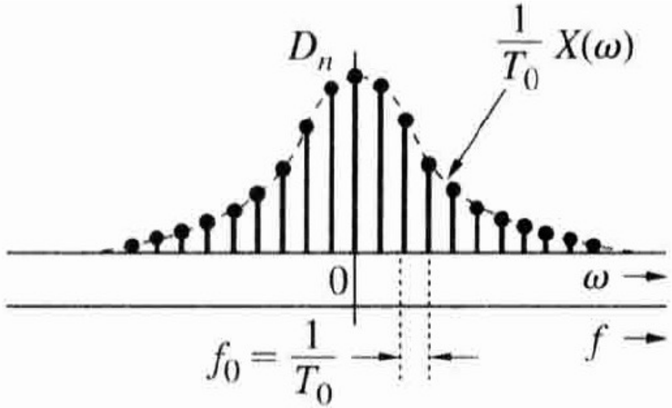
L8.4

Spectral Sampling (2)

- If we now CONSTRUCT a periodic signal $x_{T_0}(t)$, we will expect the spectrum of this signal to be discrete (expressed as Fourier series).



$\longleftrightarrow$



$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

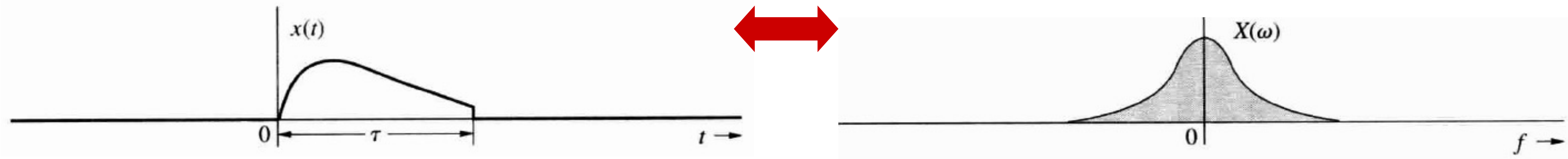
where $D_n = \dfrac{1}{T_0}\displaystyle\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt = \dfrac{1}{T_0}\displaystyle\int_0^{\tau} x(t)e^{-jn\omega_0 t}dt$

therefore $$\boxed{D_n = \frac{1}{T_0}X(n\omega_0)}$$

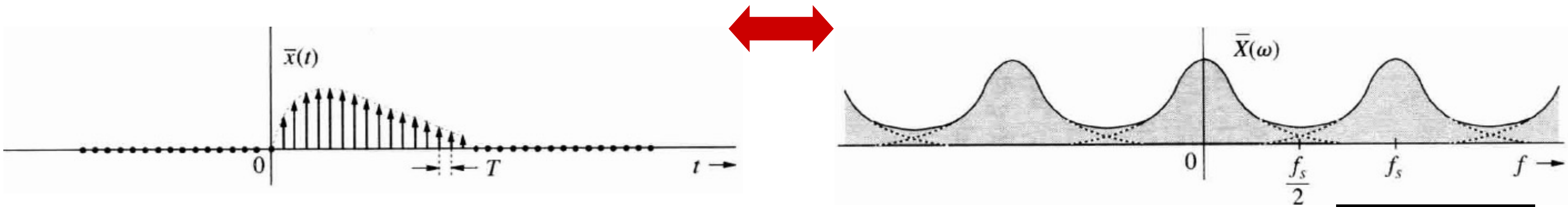
L8.4

The Discrete Fourier Transform (DFT) (1)

- Fourier transform is computed (on computers) using discrete techniques.
- Such numerical computation of the Fourier transform is known as Discrete Fourier Transform (DFT).
- Begin with time-limited signal x(t), we want to compute its Fourier Transform X(ω).



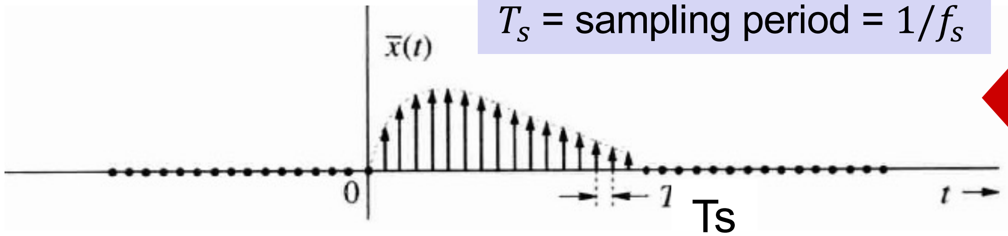
- We know the effect of sampling in time domain:



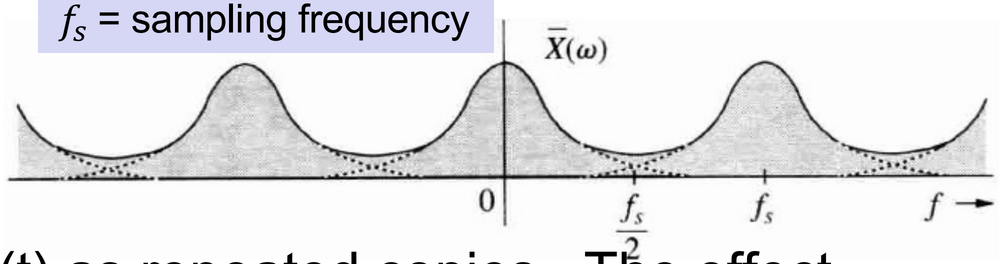
L8.5

The Discrete Fourier Transform (DFT) (2)

T_s = sampling period = $1/f_s$

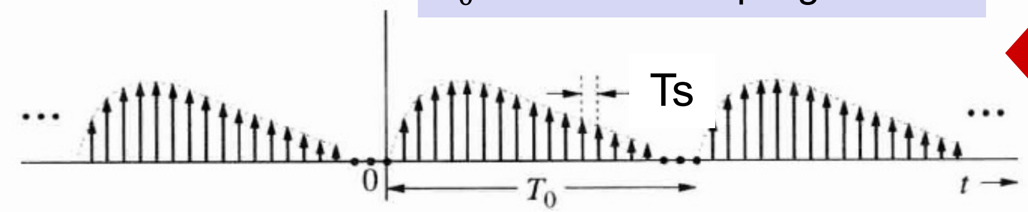


f_s = sampling frequency

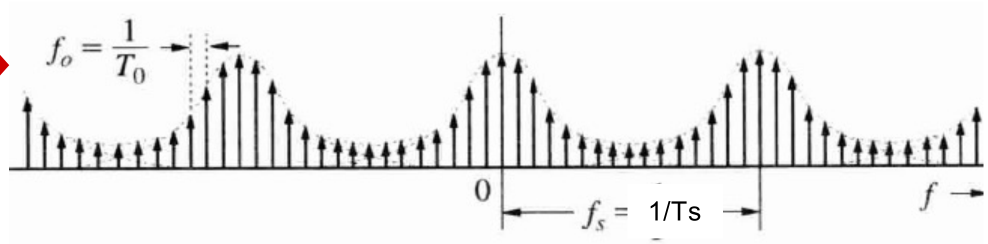


- Now construct the sampled version of x(t) as repeated copies. The effect (from slides 6-8) is sampling the spectrum.

T_0 = width of sampling window



f_0 = frequency increment = $1/T_0$



Number of time samples in T_0

$$N_0 = \frac{T_0}{T_s}$$

Number of frequency samples in f_s

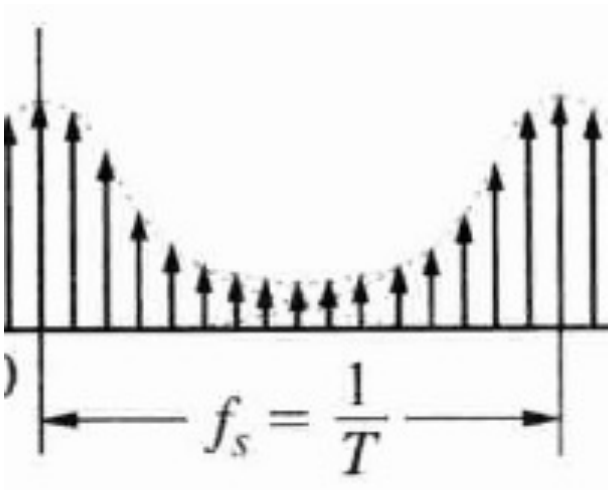
$$N'_0 = \frac{f_s}{f_0}$$

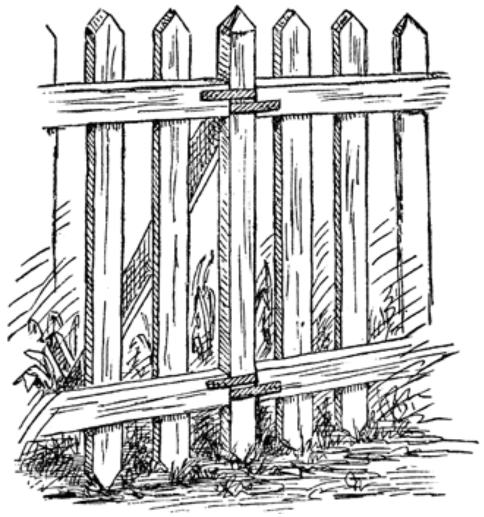
$$N_0 = \frac{T_0}{T_s} = \frac{1/f_0}{1/f_s} = \frac{f_s}{f_0} = N'_0$$

L8.5

Picket Fence Effect

- Numerical computation method yields uniform sampling values of $X(\omega)$.
- Information between samples in spectrum is missing – picket fence effect:





- Can improve spectral resolution by increasing number of samples used in the window N_0, i.e. the period of signal being transformed T_0.

L8.5

Formal definition of DFT

- If x[nT] and X[rω_0] are the nth and rth samples of x(t) and X(ω) respectively, then we define:

$$x_n = T_s \times x[nT_s] = \frac{T_0}{N_0} x[nT]$$

and

$$X_r = X(r\omega_0)$$

where

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

- Then

Forward DFT

$$X_r = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n}$$

Inverse DFT

$$x_n = \frac{1}{N_0} \sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n}$$

$$\Omega_0 = \omega_0 T = \frac{2\pi}{N_0}$$

L8.5

Parseval's Theorem

- The energy of a signal x(t) can be derived in time or frequency domain:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

- Proof:

$$x^*(t) \Longleftrightarrow X^*(-\omega)$$

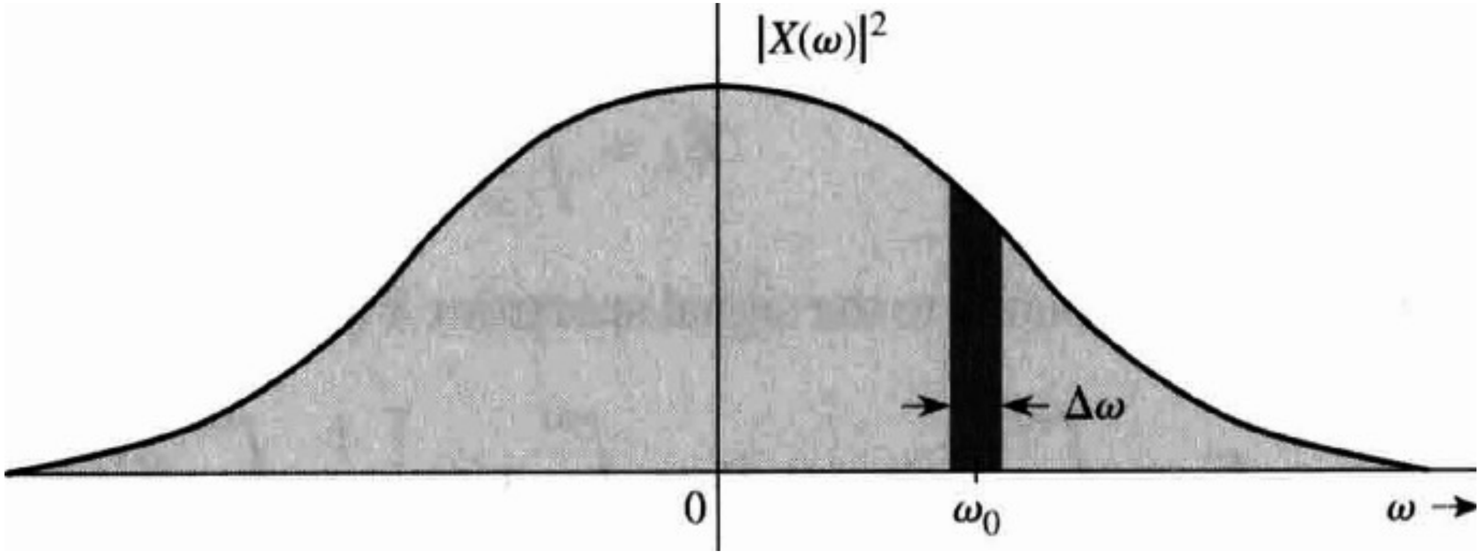
$$\begin{aligned}
E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt &= \int_{-\infty}^{\infty} x(t)x^*(t)\,dt \\
&= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}\,d\omega\right]dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega
\end{aligned}$$

Change order of integration

L7.6

Energy Spectral Density of a signal

- Total energy is area under the curve of $|X(\omega)|^2$ vs ω (divided by 2π).



- The energy over a small frequency band $\Delta\omega$ ($\Delta\omega \rightarrow 0$) is:

$$\Delta E_x = \frac{1}{2\pi}|X(\omega)|^2 \Delta\omega = |X(\omega)|^2 \Delta f \qquad \frac{\Delta\omega}{2\pi} = \Delta f \text{ Hz}$$

Energy spectral density (per unit bandwidth in Hz)

L7.6

Energy Spectral Density of a REAL signal

- If x(t) is a real signal, then $X(\omega)$ and $X(-\omega)$ are conjugate:

$$|X(\omega)|^2 = X(\omega)X^*(\omega) = X(\omega)X(-\omega)$$

- This implies that $X(\omega)$ is an even function. Therefore

$$E_x = \frac{1}{\pi}\int_0^{\infty} |X(\omega)|^2\, d\omega$$

- Consequently, the energy contributed by a real signal by spectral components between ω_1 and ω_2 is:

$$\Delta E_x = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} |X(\omega)|^2\, d\omega$$

L7.6

Example

- Find the energy E of signal x(t) = e^{-at} u(t). Determine the frequency W (rad/s) so that the energy contributed by the spectral component from 0 to W is 95% of the total signal energy E.
- Take FT of x(t):

$$X(\omega) = \frac{1}{j\omega + a}$$

- By Parseval's theorem:

$$E_x = \frac{1}{\pi}\int_0^\infty |X(\omega)|^2\, d\omega = \frac{1}{\pi}\int_0^\infty \frac{1}{\omega^2 + a^2}\, d\omega = \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\bigg|_0^\infty = \frac{1}{2a}$$

- Energy in band 0 to W is 95% of this, therefore:

$$\frac{0.95}{2a} = \frac{1}{\pi}\int_0^W \frac{d\omega}{\omega^2 + a^2} = \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\bigg|_0^W = \frac{1}{\pi a}\tan^{-1}\frac{W}{a}$$

$$\frac{0.95\pi}{2} = \tan^{-1}\frac{W}{a} \quad \Longrightarrow \quad W = 12.706a \text{ rad/s}$$

- Note: For this signal, 95% of energy is in the lower frequency band from 0 to 12.7a rad/s or 2.02a Hz!!!

L7.6

Three Big Ideas

1. Extracting a portion of a signal can be modelled by multiplying the signal with a **rectangular window**. However, the sudden changes at the window boundaries modify the signal spectrum.

2. This causes **spectral spreading** to neighbouring frequencies and leakages to higher frequencies. Both can be reduced by using other types of **window functions** such as Hamming or Hanning, which have smooth cut offs.

3. **Discrete Fourier Transform** (DFT) is used to calculate the Fourier Transform in a computer. This is done by taking the windowed portion of the signal **and construct a periodic signal** from it. The result is a sampled Fourier Transform with frequency step $f_0 = 1/T_0$, where T_0 is the window function width.